

Polynomial rings revisited

For the rest of the semester, all rings will be commutative with $1 \neq 0$.

We've seen that polynomial rings over fields are Euclidean domains (with norm $N(0)=0$, $N(f)=\deg f$, $f \neq 0$) and hence PIDs. However, if R is not a field, e.g. $R = \mathbb{Z}$, then R will have some nonzero proper ideal I , and we get the following relationship:

Prop: Let $I \subseteq R$ be an ideal, and let $(I) \subseteq R[x]$ be the ideal generated by I (i.e. polynomials w/ coefficients in I). Then

$$(R/I)[x] \cong R[x]/(I).$$

Pf: Consider the map $\varphi: R[x] \rightarrow (R/I)[x]$ defined by

$$a_0 + a_1 x + \dots + a_n x^n \mapsto \bar{a}_0 + \bar{a}_1 x + \dots + \bar{a}_n x^n, \quad \text{where } \bar{a}_i \text{ is the image of } a_i \text{ in } R/I.$$

φ is surjective, and a polynomial $f(x) = a_0 + \dots + a_n x^n$ in $R[x]$ maps to 0 $\iff \bar{a}_i = \bar{0}$ in $R/I \forall i \iff a_i \in I \forall i \iff f(x) \in (I)$.

Thus, $\ker \varphi = (I)$, so $R[x]/(I) \cong (R/I)[x]$. \square

Cor: $I \subseteq R$ is prime $\iff (I) \subseteq R[x]$ is prime.

Pf: $I \subseteq R$ is prime $\iff R/I$ is an int. domain $\iff (R/I)[x] \cong R[x]/(I)$ is an int. domain $\iff (I)$ is prime. \square

Ex: $\mathbb{Z}[x] / (n) \cong (\mathbb{Z}/n\mathbb{Z})[x]$.

We can naturally extend to polynomial rings in several variables by just adjoining an additional variable. e.g. $R[x, y] := R[x][y]$. More generally:

Def: $R[x_1, \dots, x_n]$ is defined inductively by

$$R[x_1, \dots, x_n] := R[x_1, \dots, x_{n-1}][x_n]$$

More concretely, if $a \in R$, then $a x_1^{d_1} \dots x_n^{d_n}$ is called a monomial of degree $d_1 + \dots + d_n$, and a polynomial in $R[x_1, \dots, x_n]$ is a finite sum of monomials.

The degree of a polynomial is the largest degree of a nonzero monomial summand.

Ex: In $\mathbb{Z}[x, y]$, $x^2 + xy + 3y$ is a polynomial of degree 2.